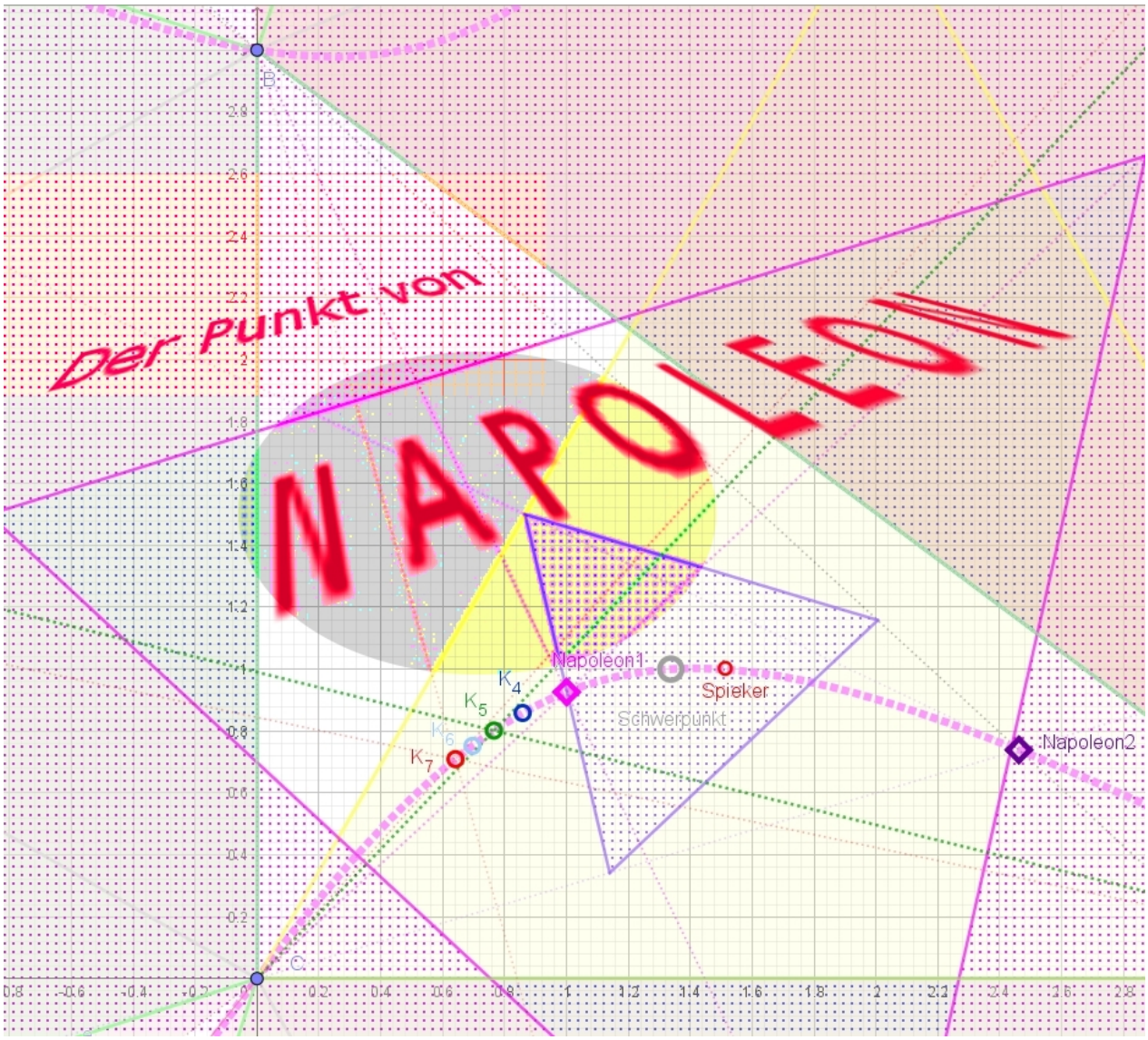
THE NAPOLEON POINT

by U. R. • English translation



The Napoleon Point

NAPOLEON

by U. R.

When I’m looking over my shoulder,

what

do U think I see?

Velvet Underground

Translator’s guide to the original diagram labels

The geometrical diagrams are reproduced from the German original. The following recurring labels will help readers interpret annotations embedded in the images.

GERMAN IN THE FIGURES	ENGLISH MEANING
Schwerpunkt	centroid
Seitenmitte	midpoint of a side
Mittelsenkrechte	perpendicular bisector
Winkelhalbierende	angle bisector
gleichseitiges Dreieck	equilateral triangle
rechtwinkliges Dreieck	right triangle
stumpfwinkliges Dreieck	obtuse triangle
Napoleon außen	outer Napoleon point
Napoleon innen	inner Napoleon point
Fläche	area
Kiepert-Hyperbel	Kiepert hyperbola
Spiekerpunkt	Spieker point
Quadrat	square
reguläres n-Eck	regular n-gon

Symbols, point names, and numerical data in the figures retain their original notation.

Imagine that “Geometria” were a woman who said of herself:

“I am made of mathematics!”

More precisely, I emerged from algebra and arithmetic!

I am the perfectly straight path that never ends: I live forever!

Do I fascinate you? And if so, do you love me, or would you rather have nothing at all to do with me?

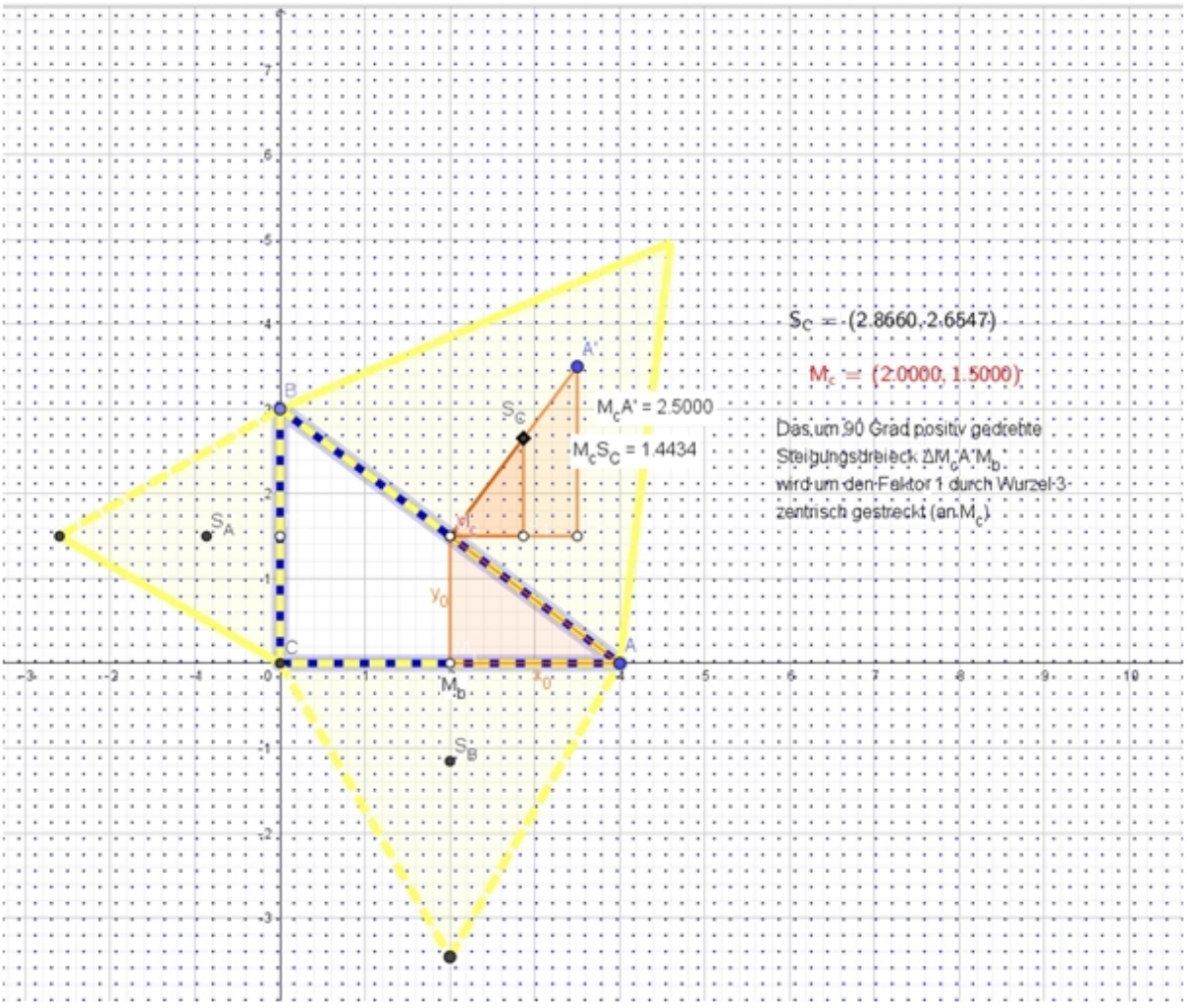
And if you wanted to ask me a question, what would that question be?

Or may I, instead, ask you a question:

When I look over my shoulders, what do you think it is that I see?

Well, it is strange; it is very, very, very strange!

As we shall see, Napoleon Bonaparte was fascinated by her, too!

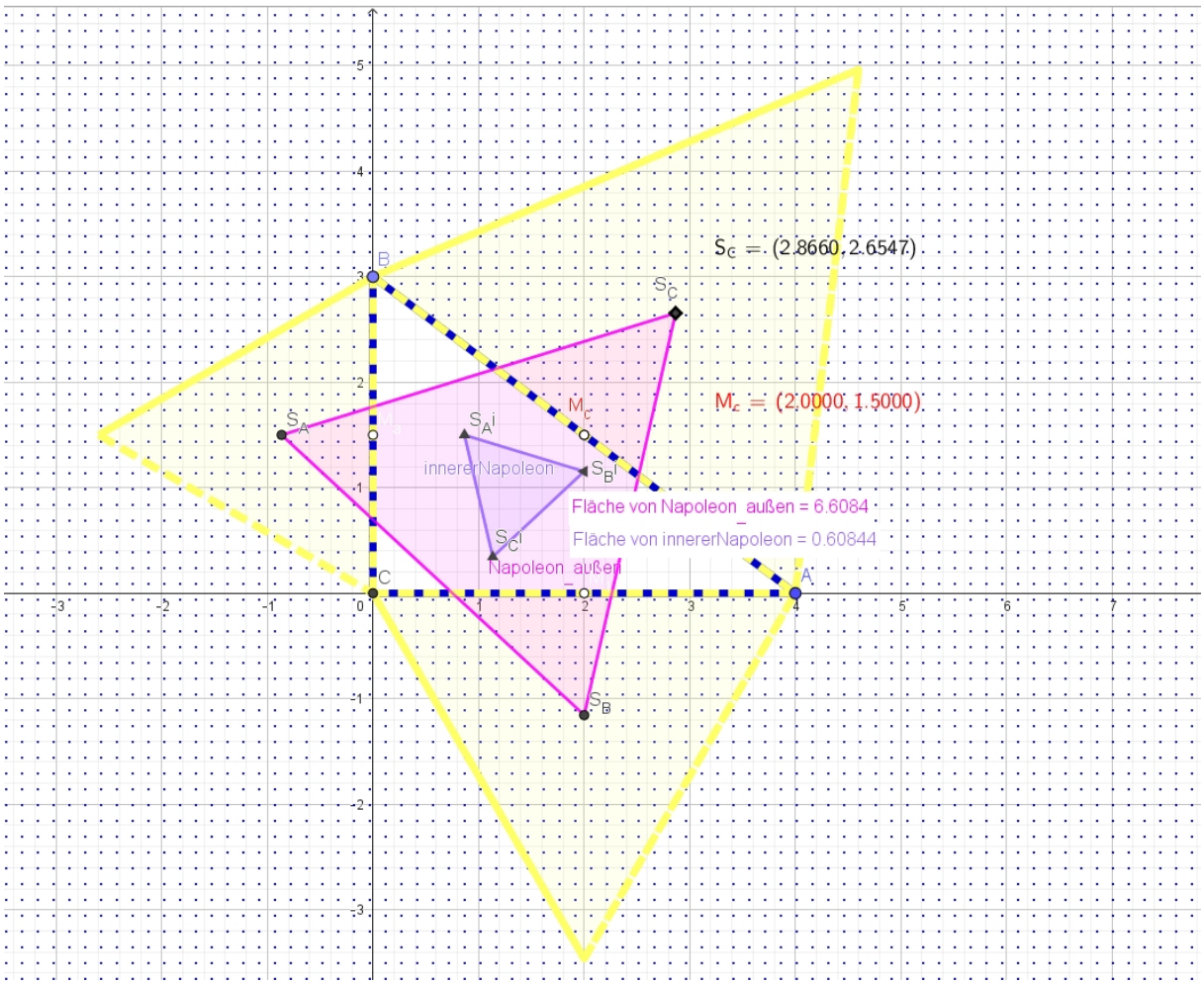


To locate the centroid of a square constructed on one of the three sides of a triangle, simply move perpendicularly from the midpoint of the side by half its length. For an equilateral triangle constructed on a side, the centroid is only one third of its altitude h = ½√3 a above the base; therefore half the side length, ½a, must be divided by √3.

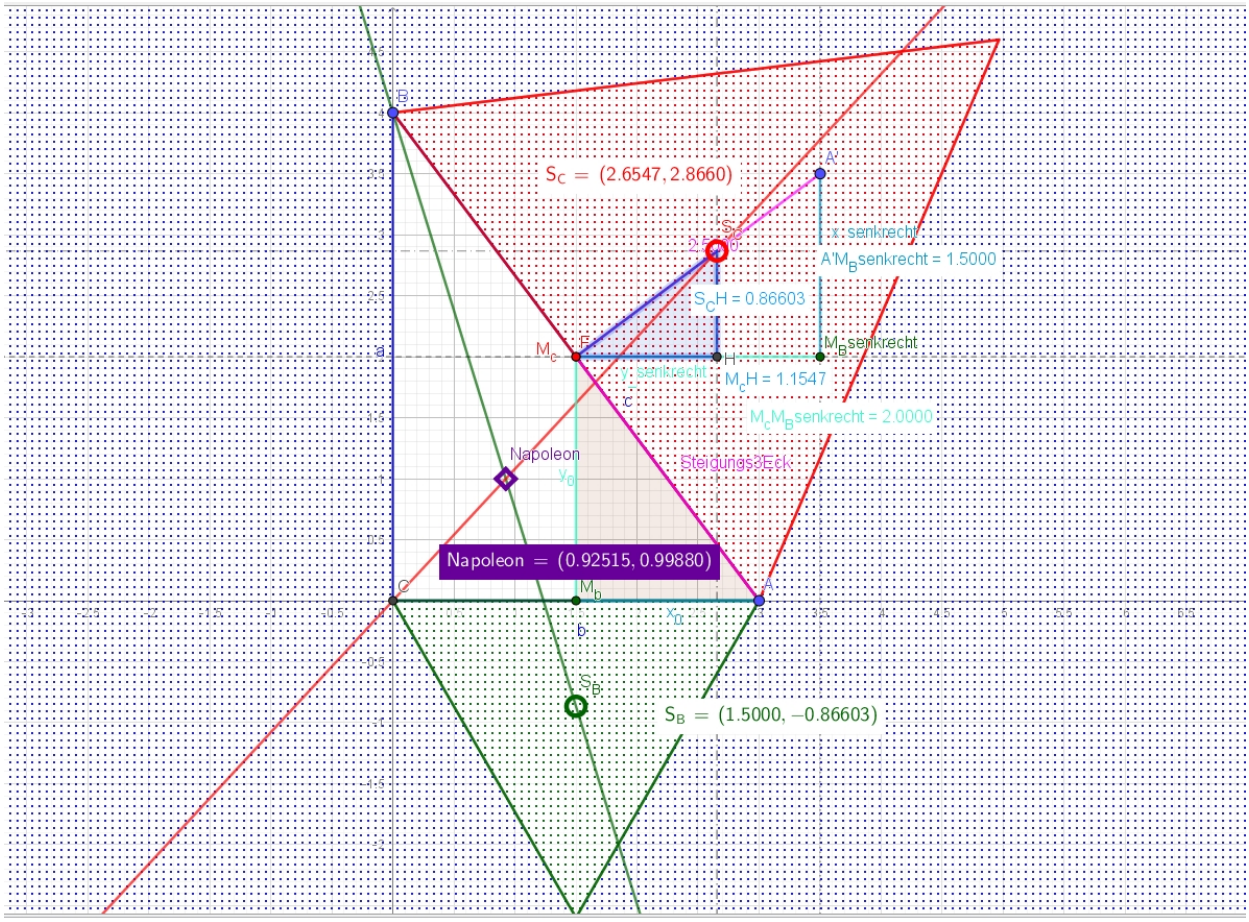
There is a central theorem of Euclidean geometry with whose help every other theorem can be proved: **the Pythagorean theorem.** The whole of Euclidean geometry stands or falls with it.

Just as squares are constructed on the sides of a right triangle in the familiar figure for that theorem, one can instead construct regular (equilateral) triangles, as Napoleon's theorem requires.

Napoleon connected their three centers (centroids) and once again found an equilateral triangle, regardless of the shape of the original triangle ΔABC!

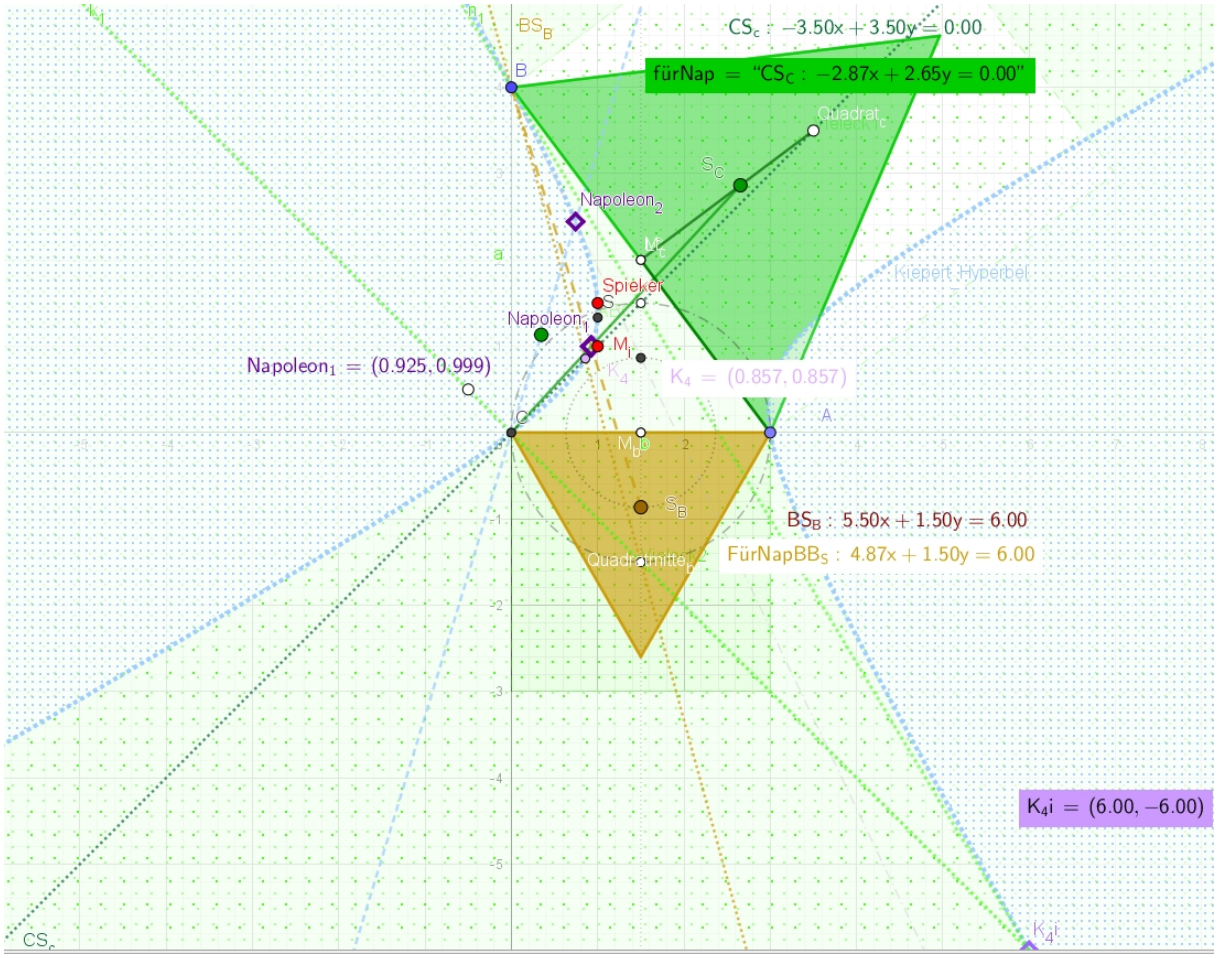


The lines joining the centroids S_A, S_B and S_C to the respective opposite vertices A, B and C intersect at one point. If the equilateral triangles are constructed externally (or internally), this is the Napoleon point (or the second Napoleon point).



On M_CS_C, the perpendicular bisector of the hypotenuse c, and on M_bS_B, the perpendicular bisector of side b, lie the centroids of all regular n-gons constructed on the corresponding sides.

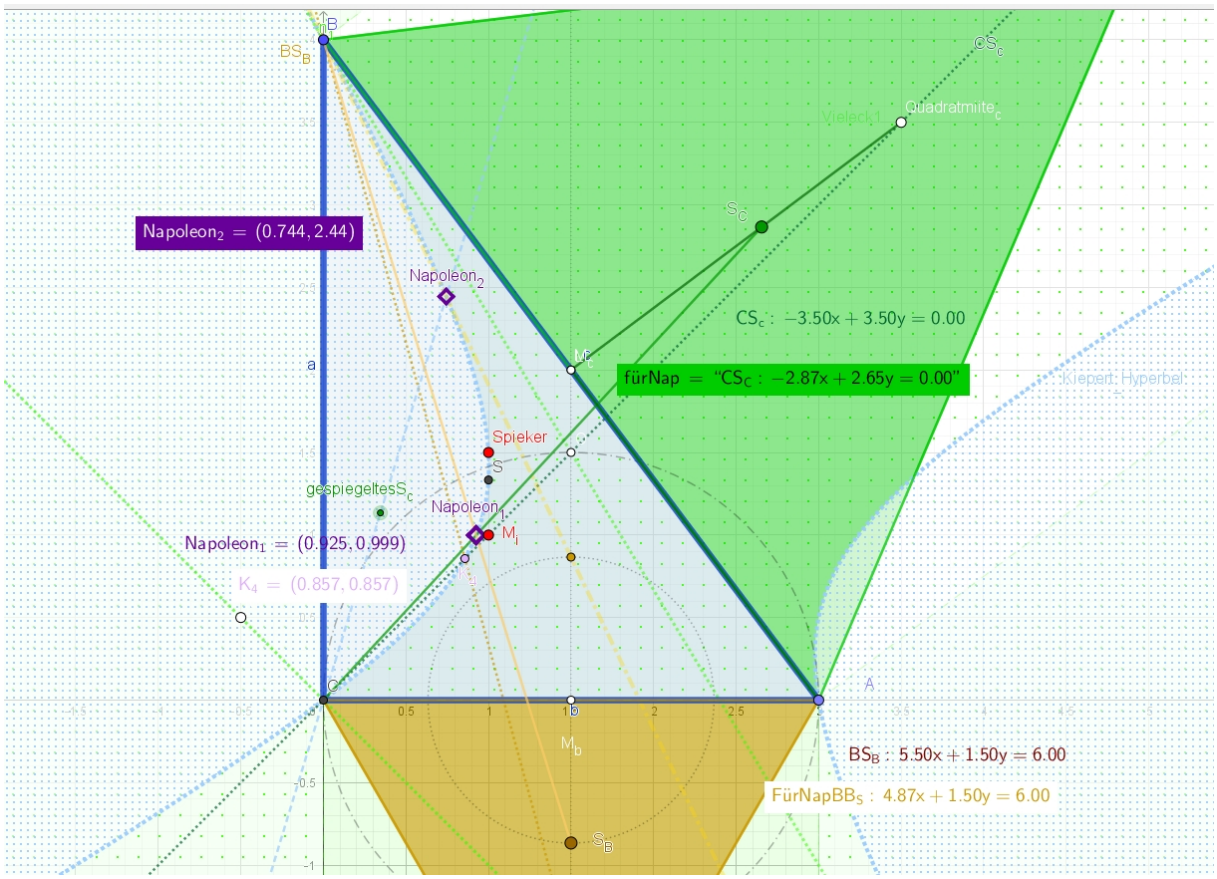
Calculating the Napoleon point for right triangles



First, let us construct the squares again (that is, $\lambda = 1$). Why do the x and y values coincide? Because the two K_4 points lie on the angle bisectors at $(0{,}875;\ 0{,}875)$ and $(6;\ -6)$!

Place the right angle at the origin C, and the points A and B on the coordinate axes.

> A = (a; 0) B = (0; b) C = (0; 0) a = 3; b = 4; c = 5



The centroid of the regular n-gon constructed on side b lies on the perpendicular bisector of b. We obtain S_B by starting at the midpoint $M_B = (½a;\ 0)$ and moving downward, in the negative y-direction, by half the side length multiplied by the factor λ. For $\lambda = 1$, this gives the centroid of the constructed square; for $\lambda = 1/\sqrt{3}$, it gives the centroid of the constructed equilateral triangle, used to construct the Napoleon point.

> $S_B = M_B - \lambda\ (0;\ ½a) = (½a;\ -½\lambda a)$

In the figure, $S_B = (1{,}5;\ -0{,}8660\ldots)$.

For the line BS_B, we use the two-point form:

> $(y - y_0)/(x - x_0) = (y_1 - y_0)/(x_1 - x_0)$ $(= m)$

with $B = (x_0; y_0) = (0; b)$ and $S_B = (x_1; y_1) = (½a; -½\lambda a)$.

$(y - b)/(x - 0) = (-½\lambda a - b)/(½a - 0) = -(½\lambda a + b)/(½a)$

$y - b = -(\lambda a + 2b\ /a)x$

> $y = -(\lambda a + 2b\ /a)x + b$

In the figure, with $a = 3$ and $b = 4$:

$4{,}87x + 1{,}5y = 6$

$y = -(1/\sqrt{3} + 8/3)x + 4$

$y = -3{,}2440\ldots x + 4$ OK

The centroid S_C is obtained by placing λ times the “slope triangle rotated by 90°” at the midpoint of side c.

> $S_C = M_C + \lambda(½b; ½a) = (½a + \lambda ½b; ½b + \lambda ½a)$

CS_C is a line through the origin, $y = mx$, with $m = y_{S_C} / x_{S_C}$; hence:

$y = x\ (½b + \lambda ½a)\ /\ (½a + \lambda ½b)$

The intersection of the red line CS_C and the green line BS_B is the Napoleon point.

> $y = -(\lambda a + 2b/a)x + b$ and $y = [(b + \lambda a)/(a + \lambda b)]\, x$

We set the two expressions equal to calculate the x-coordinate:

$(\lambda a + 2b/a)x + [(b + \lambda a)/(a + \lambda b)]x = b$

$[(\lambda a + 2b/a) + (b + \lambda a)/(a + \lambda b)]x = b$

$x[(\lambda a + 2b/a)(a + \lambda b) + (b + \lambda a))] / (a + \lambda b) = b$

> $x = b(a + \lambda b) / [(\lambda a + 2b/a)(a + \lambda b) + (b + \lambda a)]$

As λ tends to zero, we obtain the centroid S:

$x = a\, b / [(2b + (b)] = a/3 = 1$ (if $a = 3$)

$x = \{ a-1 \}/2 = 1$

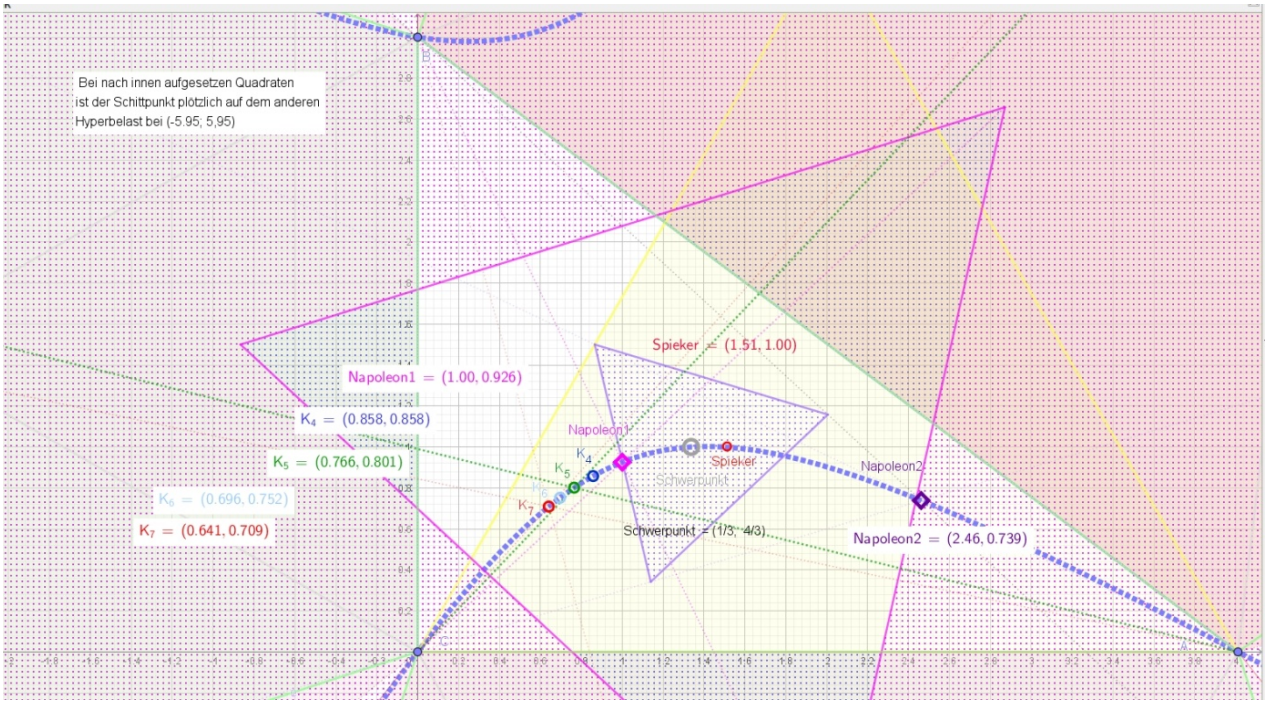
and, substituting this into

$y = -(\lambda + 2b/a)x + b$

$y = -2b/a + b = 4 - 8/3 = 4/3$

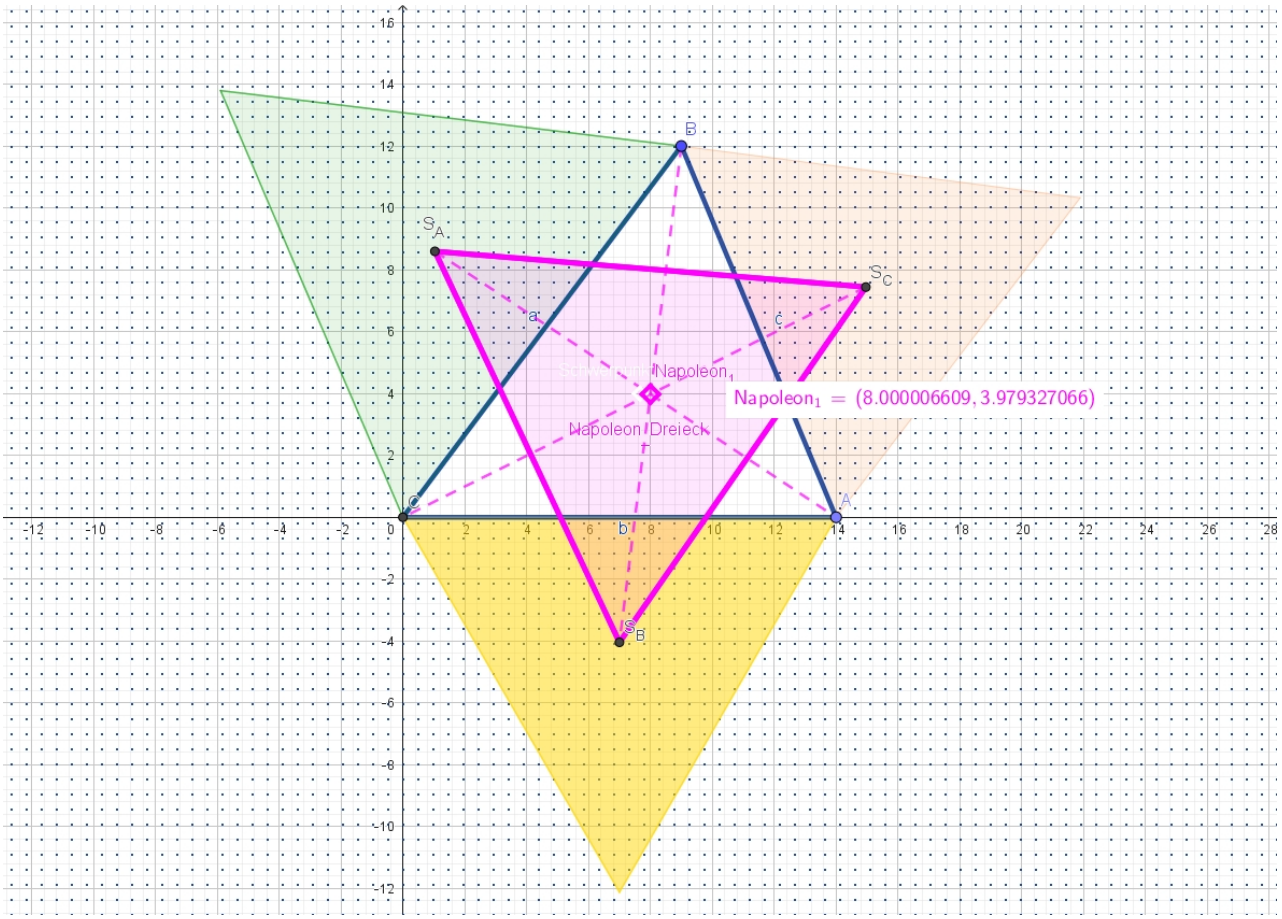
S = 1/3 (A + B + C) = 1/3 (3; 4) = (1; 1,3333…) Kanton Clarus

[1] This also follows more quickly from $y = x\,(b + \lambda a)/(a + \lambda b) = b/a = 4/3$ for $x = 1$.

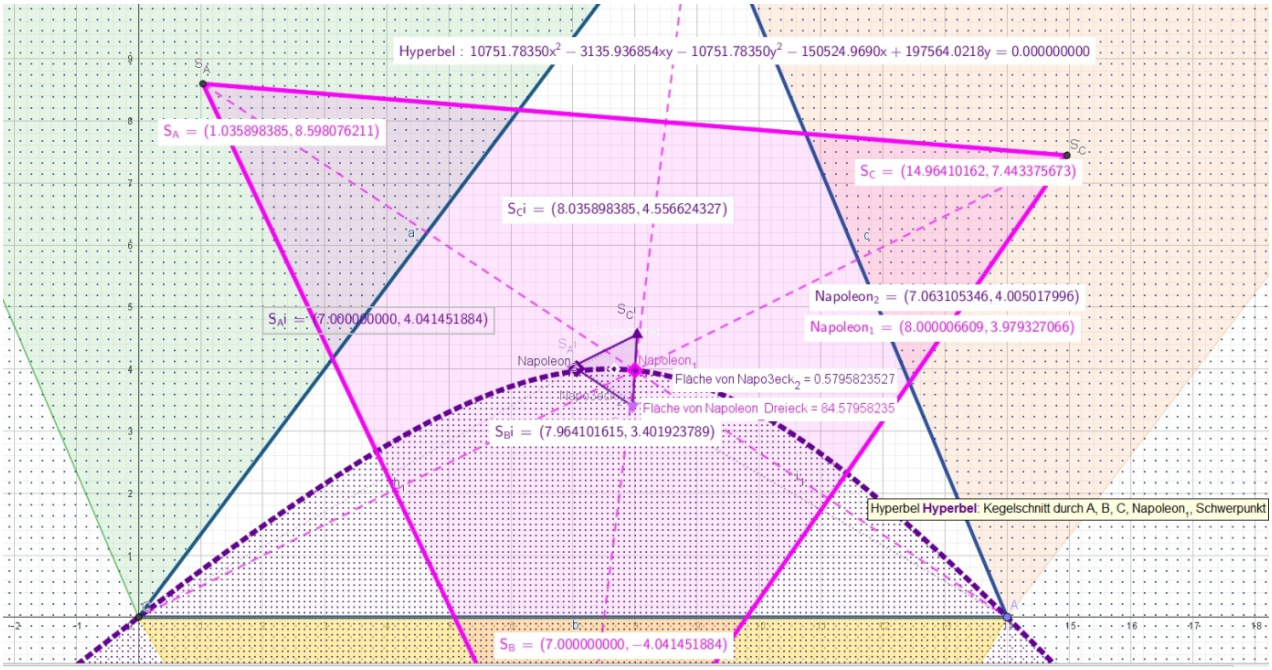


The intersection points from Napoleon (≡ K_3) to K_7, obtained by constructing regular polygons up to heptagons on the sides, lie on the Kiepert hyperbola passing through A, B, C, S and the Spieker point. The Spieker point is the image of the incenter M_i under a dilation centered at S with factor $k = -½$.

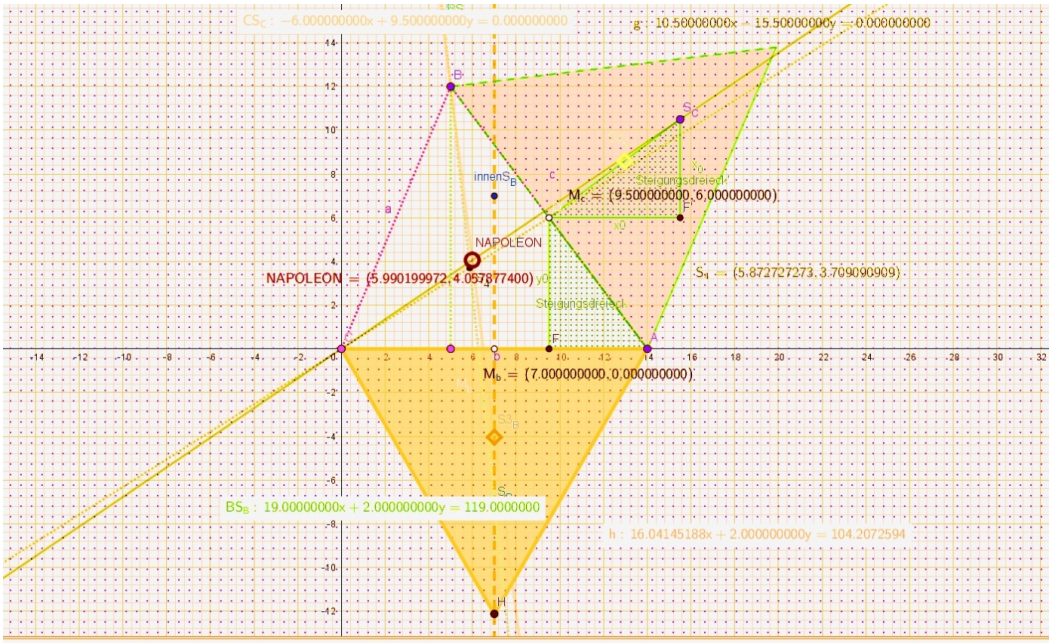
Napoleon’s theorem for non-right triangles



The second standard triangle, with side lengths 13, 14 and 15.



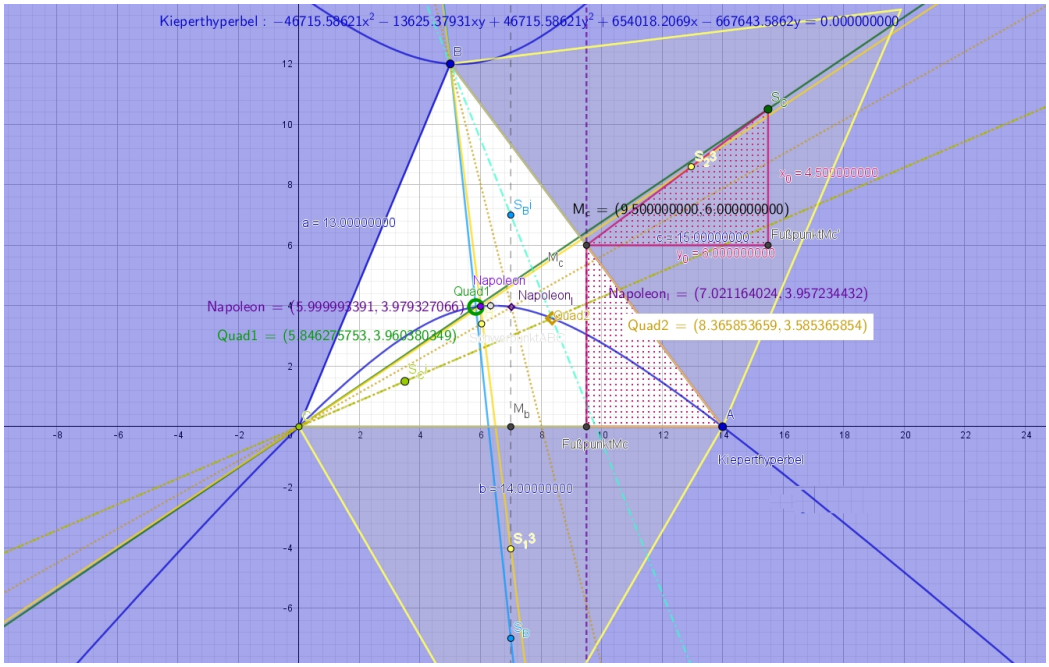
The two Napoleon triangles have the same centroid as the original triangle ABC. The area of the Napoleon triangle formed by the centroids of the equilateral triangles constructed externally is always greater than the area of the second Napoleon triangle, formed using internally constructed equilateral triangles, by exactly the area of triangle ABC. Both Napoleon points lie on the Kiepert hyperbola and are separated by the (white) centroid S.



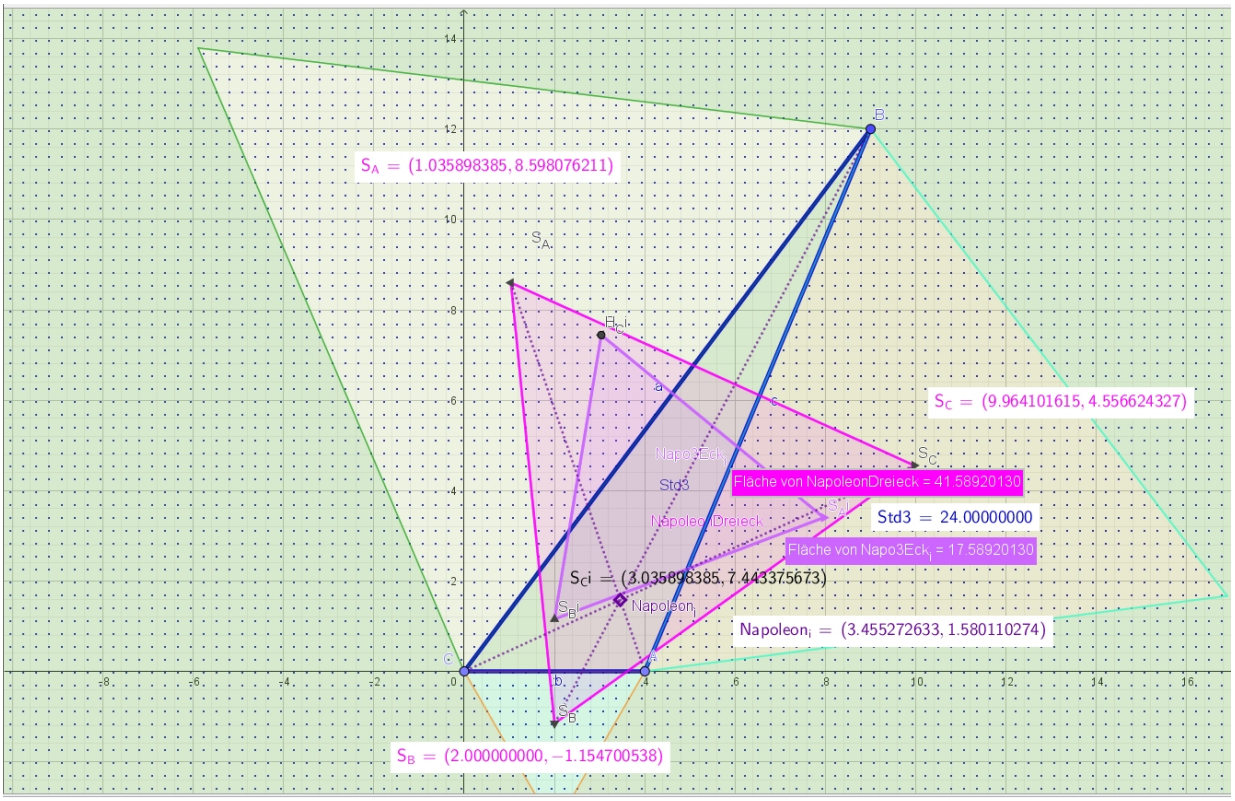
Instead of drawing the regular triangles (or the yellow squares), it is sufficient to use two of their centroids, S_B and S_C.

The midpoint of side b is $M_b = (½a; 0)$, and the midpoint of side c is:

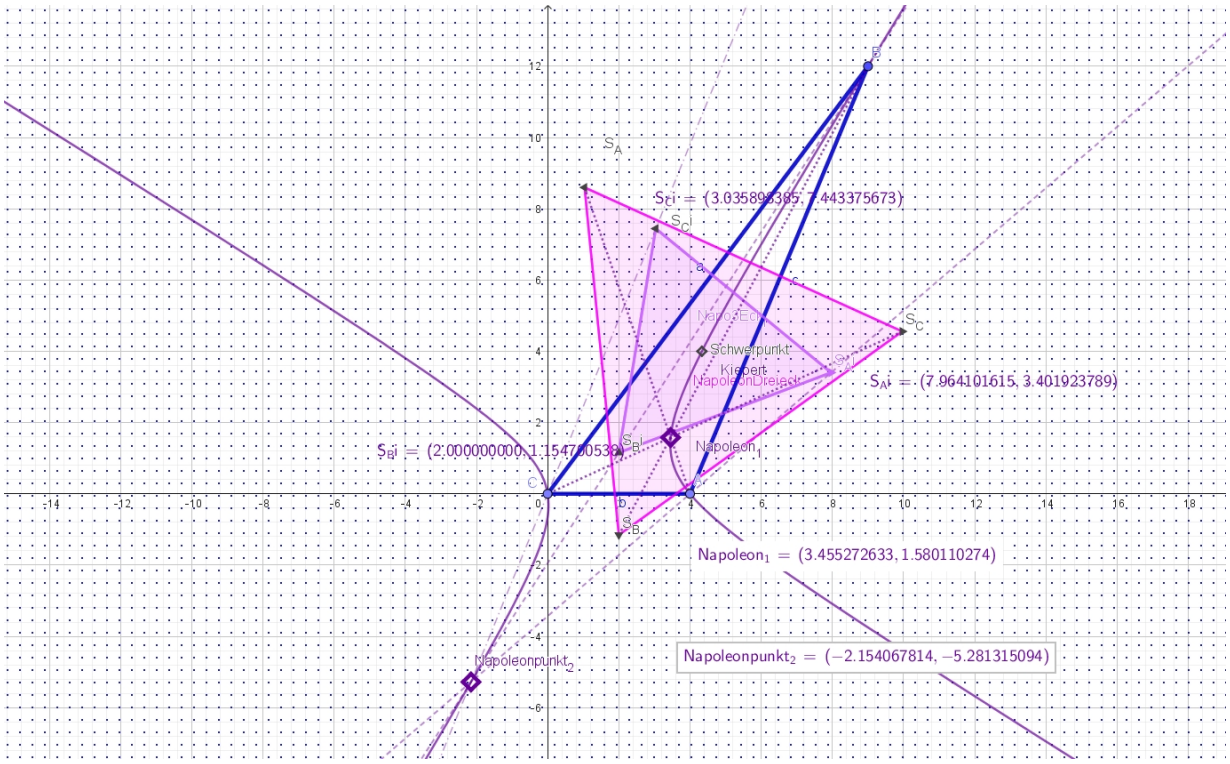
$M_c = ½(a + b_x; b_y) = (½a + ½b_x; ½b_y)$, $2M_c = (a + b_x; b_y)$



Obtuse triangles



The third standard triangle.


$S_C = (3.035894385, 7.443375673)$
$S_A = (7.964101615, 3.401923789)$
$S_B = (2.000000000, 1.154700538)$
Napoleon$_1$ = (3.455272633, 1.580110274)
Napoleonpunkt$_2$ = (−2.154067814, −5.281315094)
Schwerpunkt
Kiepert
NapoleonDreieck
Napoleon$_1$
Napoleonpunkt$_2$